# The connection between Cox (PH) modelling and the log-rank test

Assuming no ties at $t_{(j)}$, $j = 1, \ldots, r$, $d_j = d_{1j} + d_{2j} = 1$, $x_i = 1$ (group 1) or 0 (group 2)

$$h_i(t) = e^{\beta x_i} h_0(t),$$

$$L(\beta) = \prod_{j=1}^{r} \frac{\exp(\beta x_{(j)})}{\sum_{l=1}^{n_j} \exp(\beta x_l)},$$

Then (since $n_j = n_{1j} + n_{2j}$),

$$\log L(\beta) = \sum_{j=1}^{r} \beta x_{(j)} - \sum_{j=1}^{r} \log\left\{ \sum_{l=1}^{n_j} \exp(\beta x_l) \right\}.$$

$$\sum_{l-1}^{n_j} \exp(\beta x_l) = n_{1j} e^{\beta} + n_{2j},$$

$$\log L(\beta) = d_1 \beta - \sum_{j=1}^{r} \log\{ n_{1j} e^{\beta} + n_{2j} \}.$$

Where

$$d_1 = \sum_{j=1}^{r} d_{1j}$$

$$\frac{\partial \log L(\beta)}{\partial \beta} = \sum_{j=1}^{r} \left( d_{1j} - \frac{n_{1j} e^{\beta}}{n_{1j} e^{\beta} + n_{2j}} \right),$$

$$\begin{aligned} \frac{\partial^2 \log L(\beta)}{\partial \beta^2} &= -\sum_{j=1}^{r} \frac{(n_{1j} e^{\beta} + n_{2j}) n_{1j} e^{\beta} - (n_{1j} e^{\beta})^2}{(n_{1j} e^{\beta} + n_{2j})^2} \\ &= -\sum_{j=1}^{r} \frac{n_{1j} n_{2j} e^{\beta}}{(n_{1j} e^{\beta} + n_{2j})^2}. \end{aligned}$$

$$u(0) = \sum_{j=1}^{r} \left( d_{1j} - \frac{n_{1j}}{n_{1j} + n_{2j}} \right),$$

$$i(0) = \sum_{j=1}^{r} \frac{n_{1j} n_{2j}}{(n_{1j} + n_{2j})^2}.$$

The score test statistic under the null hypothesis that $\beta = 0$, $u^2(0)/i(0)$ has a chi-square distribution with one degree of freedom.

Recall (since $d_j = 1$, thus $n_j - d_j = n_j - 1$, $d_j(n_j - d_j) = (n_j - 1)$):

$$v_{1j} = \frac{n_{1j} n_{2j} d_j (n_j - d_j)}{n_j^2 (n_j - 1)},$$

$$\text{var}(U_L) = \sum_{j=1}^{r} v_{1j} = V_L,$$

Thus, $u^2(0)/i(0) = U^2_L/V_L$.