The connection between Cox (PH) modelling and the log-rank test

Assuming no ties at $t_{(j)}$, j = 1, ..., r, $d_j = d_{1j} + d_{2j} = 1$, $x_i = 1$ (group 1) or 0 (group 2)

$$h_i(t) = e^{eta x_i} h_0(t),$$

$$L(eta) = \prod_{j=1}^r rac{\exp(eta x_{(j)})}{\sum\limits_{l=1}^{n_j} \exp(eta x_l)},$$

Then (since $n_j = n_{1j} + n_{2j}$),

$$egin{aligned} \log L(eta) &= \sum_{j=1}^r eta x_{(j)} - \sum_{j=1}^r \logiggl\{\sum_{l=1}^{n_j} \exp(eta x_l)iggr\}. \ &\sum_{l=1}^{n_j} \exp(eta x_l) = n_{1j} e^eta + n_{2j}, \end{aligned}$$

$$\log L(eta) = d_1eta - \sum_{j=1}^r \logig\{n_{1j}e^eta + n_{2j}ig\}.$$

Where

$$d_1 = \sum_{j=1}^r d_{1j}$$

$$rac{\partial \log L(eta)}{\partial eta} = \sum_{j=1}^r \Biggl(d_{1j} - rac{n_{1j} e^eta}{n_{1j} e^eta + n_{2j}} \Biggr),$$

$$egin{aligned} rac{\partial^2 \log L(eta)}{\partial eta^2} &= -\sum_{j=1}^r rac{(n_{1j}e^eta+n_{2j})n_{1j}e^eta-(n_{1j}e^eta)^2}{(n_{1j}e^eta+n_{2j})^2} \ &= -\sum_{j=1}^r rac{n_{1j}n_{2j}e^eta}{(n_{1j}e^eta+n_{2j})^2}. \end{aligned}$$

$$u(0) = \sum_{j=1}^r igg(d_{1j} - rac{n_{1j}}{n_{1j} + n_{2j}} igg),$$

$$i(0) = \sum_{j=1}^r rac{n_{1j} n_{2j}}{\left(n_{1j} + n_{2j}
ight)^2}.$$

The score test statistic under the null hypothesis that $\beta = 0$, $u^2(0)/i(0)$ has a chisquare distribution with one degree of freedom.

Recall (since $d_j = 1$, thus $n_j - d_j = n_j - 1$, $d_j(n_j - d_j) = (n_j - 1)$):

$$v_{1j} = rac{n_{1j}n_{2j}d_j(n_j-d_j)}{n_j^2(n_j-1)},$$

$$ext{var}\left(U_L
ight) = \sum_{j=1}^r v_{1j} = V_L,$$

Thus, $u^2(0)/i(0) = U^2_L/V_L$.